Pergamon

Int. J. Solids Structures Vol. 33, No. 3, pp. 399–417, 1996

0020–7683/96 $9.50 + .00

0020–7683(95)00040–2

INFLUENCE OF THE INTERMEDIATE MATERIAL ON THE ORDER OF STRESS SINGULARITY IN THREE-PHASE BONDED STRUCTURE

TADANOBU INOUE and HIDEO KOGUCHI†
Department of Mechanical Engineering, Nagaoka University of Technology, 1603-1 Kamitomioka, Nagaoka, Niigata 940-21, Japan

(*Received* 25 *March* 1994; *in revised form* 26 *January* 1995)

Abstract—The plane problem for dissimilar materials composed of three isotropic homogeneous wedges with arbitrary angles under surface tractions is analysed using the theory of elasticity. The order of singularity in stress fields near the apex of a three-phase bonded structure is investigated. It is first demonstrated how the order of stress singularity varies with several combinations of material with different properties in cases where the angles of wedge are of $\pi/3$–$\pi/3$–$\pi/3$ and $\pi/2$–$\pi/2$–$\pi/2$. When the materials and wedge angles of the side regions are identical, the order of stress singularity is additionally studied in detail, with the wedge angle of the intermediate region varying in cases where the total angle of the bonded wedge geometry is fixed and not fixed. A method reducing the stress singularity in a two-phase bonded structure is finally proposed.

1. INTRODUCTION

Many investigations on stress singularities have been carried out [Williams (1952), Bogy (1968, 1970, 1971a, b), Hein and Erdogan (1971), and others]. Williams (1952), for single material wedge subjected to extension under different boundary conditions (free–free, fixed–free and fixed–fixed), showed the variation of the minimum real part of the eigenvalue with the vertex angle. He demonstrated that a stress singularity occurs with vertex angles between 180° and 360° in all boundary conditions. Bogy (1968, 1970) studied stress fields in bonded elastic quarter planes under surface tractions. He found that stress fields are of the order $r^{-\lambda}$, where $0 \leqslant \lambda \leqslant 0.41$, under certain conditions. Bogy (1970) also clarified the conditions of the loadings and the material combinations in which the singularity was either logarithmic or nonexistent. Furthermore, Bogy (1971a, b) derived an eigen equation to obtain the order of stress singularity near the apex in a two-phase bonded structure with arbitrary wedge angles and investigated the stress singularity near the crack tip terminating at the interface of a bimaterial composite by using the two Dundurs composite parameters α and β (Dundurs, 1969). Dempsey and Sinclair (1979) presented two systematic methods of expanding the determinant for the N-material wedges, and Theocaris (1974) provided a formulation for the case of a full plane composed of many elastic wedges with arbitrary angles. Pageau *et al.* (1994) examined the order of stress singularity for all perfectly bonded or a disbonded two- and three-material junctions using Theocaris' formulation provided for N-materials. In all cases, the singularity depends on the wedge angles and the elastic properties of materials.

Koguchi *et al.* (1995), derived an eigen equation for investigating the order of stress singularity near the apex in a three-phase bonded structure composed of three isotropic homogeneous wedges with arbitrary angles using the Airy stress function through the Mellin transform (Sneddon, 1951; Davis, 1984). Then, four Dundurs composite parameters (Dundurs, 1969), α_{12}, β_{12} for materials 1 and 2 and α_{23}, β_{23} for materials 2 and 3, were introduced for representing six elastic constants for the bonded structure with three pairs (G_1, ν_1), (G_2, ν_2) and (G_3, ν_3) of shear modulus and Poisson's ratio. It was concluded that the order of stress singularity could be reduced by a suitable arrangement of the bonded order of materials in the three-phase bonded structure.

†To whom correspondence should be addressed.

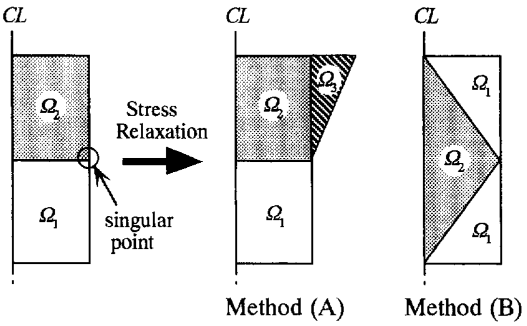


Fig. 1. Two methods for reducing the order of stress singularity for a two-phase bonded structure.

The order of stress singularity can be reduced by varying the elastic properties and the wedge angles of two materials in a two-phase bonded structure. We consider that the reduction of the order of singularity in the two-phase bonded structure will be achieved by employing methods (A) and (B) shown in Fig. 1. So, we examine how the order of stress singularity varies with combinations of the material properties and the wedge angles in methods (A) and (B). In particular, since the elastic property of the intermediate material and its wedge angle greatly influence the magnitude of the order of stress singularity in the three-phase bonded structure, the emphasis here is placed on investigating the order of singularity in the variation of the elastic property and the wedge angle of the intermediate material.

In this paper, we first consider the cases shown in Fig. 2, where the third region with wedge angles φ_3 of $\pi/3$ and $\pi/2$ is bonded to the two-phase structure with wedge angles of $(\varphi_1, \varphi_2) = (\pi/3\text{–}\pi/3)$ and $(\pi/2\text{–}\pi/2)$, where φ_1 is the wedge angle of the first region and φ_2 is

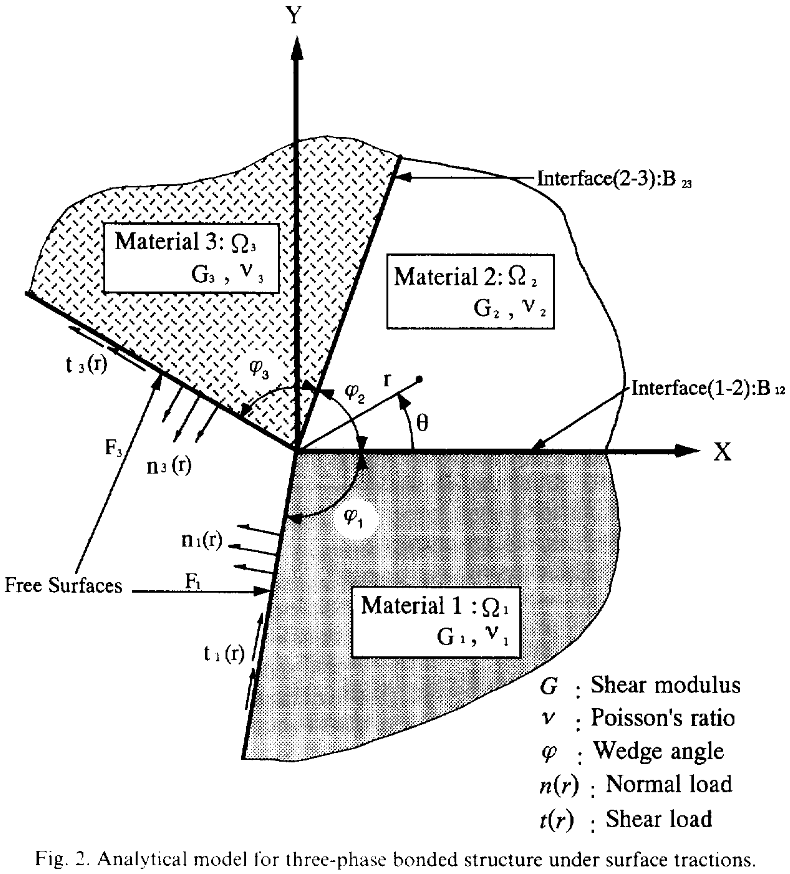


Fig. 2. Analytical model for three-phase bonded structure under surface tractions.

that of the second region. Variations of the order of stress singularity are shown on the α_{23}–β_{23} plane for several material combinations. Secondly, the case where materials of both side regions are identical is examined. The relation between the wedge angle φ_2 of intermediate material and the order of stress singularity is investigated in cases where the sum of bonded wedge angles is fixed ($\varphi_1+\varphi_2+\varphi_3$ = fixed) and where the wedge angles φ_1 and φ_3 of both side regions are equal and fixed ($\varphi_1 = \varphi_3$ = fixed). In addition, a relation between the wedge angles of both side regions and the order of singularity is examined in cases where $\varphi_1+\varphi_2+\varphi_3$ = fixed and φ_2 = fixed. Results of Williams (1952) for a single wedge subjected to extension under several boundary conditions, and those of Bogy (1971a, b) for two-phase bonded structure subjected to surface tractions and for problems involving a crack in a two-phase bonded full plane agree well with the results in our investigations.

A method for reducing the stress singularity in a two-phase bonded structure is proposed by utilizing the results obtained for the three-phase bonded structure. Furthermore, a example applying the method to a copper–alumina composite is presented.

2. DERIVATION OF THE EIGEN EQUATION

Since the derivation of the eigen equation for the three-phase bonded structure is described in detail in a previous paper (Koguchi *et al.*, 1995), only essential results used hereafter are shown in this section.

Constant m_δ is denoted by Poisson's ratio ν_δ as follows:

$$m_\delta = \begin{cases} 4(1-\nu_\delta) & \text{for plane strain} \\ 4/(1+\nu_\delta) & \text{for plane stress} \end{cases} \quad (\delta = 1,2,3). \tag{1}$$

Four Dundurs composite parameters: α_{12} and β_{12} for materials 1 and 2, and α_{23} and β_{23} for materials 2 and 3, are used in the eigen equation with six elastic constants.

$$\alpha_{12} = \frac{k_{12}m_2-m_1}{k_{12}m_2+m_1}, \quad \beta_{12} = \frac{k_{12}(m_2-2)-(m_1-2)}{k_{12}m_2+m_1}$$
$$\alpha_{23} = \frac{k_{23}m_3-m_2}{k_{23}m_3+m_2}, \quad \beta_{23} = \frac{k_{23}(m_3-2)-(m_2-2)}{k_{23}m_3+m_2}, \tag{2}$$

where

$$k_{12} = \frac{G_1}{G_2}, \quad k_{23} = \frac{G_2}{G_3}. \tag{3}$$

The subscript of elastic constants represents each region. The complex variable p is defined by

$$p = -s-1 = \xi+i\eta, \tag{4}$$

where s is a parameter of the Mellin transform, ξ is the real part of p-Re(p) and η is the imaginary part of p-Im(p).

The eigen equation for the three-phase bonded structure deduced by expanding and arranging the determinant (12×12) of the coefficients in the simultaneous equations obtained from the boundary conditions of stress and displacement can be explicitly expressed as follows:

$$\begin{aligned}\mathbf{S}(\varphi_1,\varphi_2,\varphi_3,\alpha_{12},\beta_{12},\alpha_{23},\beta_{23};p) &= \mathbf{A}(\varphi_1,\varphi_2,\varphi_3,\alpha_{12},\beta_{12},\alpha_{23},\beta_{23};p) \\ &+\mathbf{B}(\varphi_1,\varphi_2,\varphi_3,\alpha_{12},\beta_{12},\alpha_{23},\beta_{23};p)+\mathbf{C}(\varphi_1,\varphi_2,\varphi_3,\alpha_{12},\beta_{12},\alpha_{23},\beta_{23};p) \\ &+\mathbf{D}(\varphi_1,\varphi_2,\varphi_3;p).\end{aligned} \tag{5}$$

This equation (5) will generally satisfy the following relation.

$$\mathbf{S}(\varphi_1, \varphi_2, \varphi_3, \alpha_{12}, \beta_{12}, \alpha_{23}, \beta_{23}; p) = \mathbf{S}(\varphi_3, \varphi_2, \varphi_1, -\alpha_{23}, -\beta_{23}, -\alpha_{12}, -\beta_{12}; p). \quad (6)$$

The order of stress singularity is generally given by the real part of the root of $\mathbf{S}(\varphi_1, \varphi_2, \varphi_3, \alpha_{12}, \beta_{12}, \alpha_{23}, \beta_{23}; p_1) = 0$ existing within $0 < \mathrm{Re}(p) < 1$ (stress σ has the relation of $\sigma \propto r^{p-1}$ with root p). Otherwise, the root with logarithmic singularity (Bogy, 1970, 1971a) is obtained from satisfying the following equation:

$$\frac{\mathrm{d}}{\mathrm{d}p}\mathbf{S}(\varphi_1, \varphi_2, \varphi_3, \alpha_{12}, \beta_{12}, \alpha_{23}, \beta_{23}; p_1)\bigg|_{p_1 \to 1} = 0. \quad (7)$$

The root will be represented as p_1. Furthermore, when the eigen equation has multiple roots within $0 < \mathrm{Re}(p) < 1$, the root having the smallest real part will be chosen as p_1.

3. NUMERICAL RESULTS

3.1. *Loci for the root on the α_{23}–β_{23} plane*

The influence of the bonded wedge angle on the stress singularity is investigated in two cases of angles $\pi/3$–$\pi/3$–$\pi/3$ and $\pi/2$–$\pi/2$–$\pi/2$. Elastic constants for each material are within the following ranges.

$$0 < G_1, G_2, G_3 < \infty, \quad 0 \leqslant \nu_1, \nu_2, \nu_3 \leqq 0.5. \quad (8)$$

The values of Dundurs parameters, α_{12}, β_{12}, α_{23} and β_{23}, defined in eqn (2) (see Fig. 3) fall within a parallelogram (Bogy, 1970) with the values of α_{12}, α_{23} between -1 and 1. Since the domain corresponding to plane stress is included within that corresponding to plane strain, the results of plane stress are not shown here.

After a fixed point on the α_{12}–β_{12} plane from the elastic properties of materials 1 and 2 is assigned, the loci for root p_1 are drawn on the α_{23}–β_{23} plane. That is, root p_1 in the case where material 3 is bonded to the two-phase bonded structure of materials 1 and 2 is examined. In the present study, roots having relationships of $(\alpha_{12}, \beta_{12}) = (0.875, 0.25)$ and $(-0.875, -0.25)$ are searched; these will be referred to as cases 1 and 2. Also, the roots obtained for the two-phase bonded structure of materials 1 and 2 are plotted as the loci for p'_1 on the α_{12}–β_{12} plane.

Figures 4 and 6 show the loci for p'_1 plotted for the two-phase bonded structures of wedge angles $\pi/3$–$\pi/3$ and $\pi/2$–$\pi/2$, respectively. Figures 5 and 7 show the loci for p_1 for the three-phase bonded structures $\pi/3$–$\pi/3$–$\pi/3$ and $\pi/2$–$\pi/2$–$\pi/2$, respectively. In Figs 4–7, $p_1 \to 1.00$ ($p'_1 \to 1.00$) noted in the α_{23}–β_{23} (α_{12}–β_{12}) plane represents the loci for the root defined by eqn (7) exhibiting logarithmic singularity.

3.2. *Roots for bonded wedge with $\varphi_1 = \varphi_3$ and $\varphi_1 + \varphi_2 + \varphi_3$ = fixed*

It is demonstrated here how the order of stress singularity varies with the wedge angle φ_2 with the condition $(\alpha_{12}, \beta_{12}, \alpha_{23}, \beta_{23}) = (\alpha, \beta, -\alpha, -\beta)$, i.e. the mechanical properties of materials 1 and 3 are identical. The stress singularity is investigated under the conditions $(\alpha_{12}, \beta_{12}, \alpha_{23}, \beta_{23}) = (0.85, 0.30, -0.85, -0.30)$ and $(-0.85, -0.30, 0.85, 0.30)$. The former is referred to case I and the latter to case II. Case I is the condition that the bonded structure has the relation $G_1 = G_3 \gg G_2$, and case II has the relation $G_1 = G_3 \ll G_2$. We suppose that the sum of the wedge angle is fixed and the angle φ_1 is equal to φ_3. Roots in three configurations: half-planes, three-quarters-planes and full-planes are searched in detail. The variations of all the roots p existing within $0 < \mathrm{Re}(p) < 1$ with the wedge angle φ_2 are shown in Figs 8–10. Solid and broken lines represent real and imaginary parts of complex roots, respectively. In Figs 8–10, the left end ($\varphi_2 = 0°$) corresponds to a free–free single wedge with angle $\varphi_1 + \varphi_3$, and the right end ($\varphi_1 + \varphi_3 = 0°$) with angle φ_2. The roots then agree with the ones obtained from eqn (15) of Williams (1952).

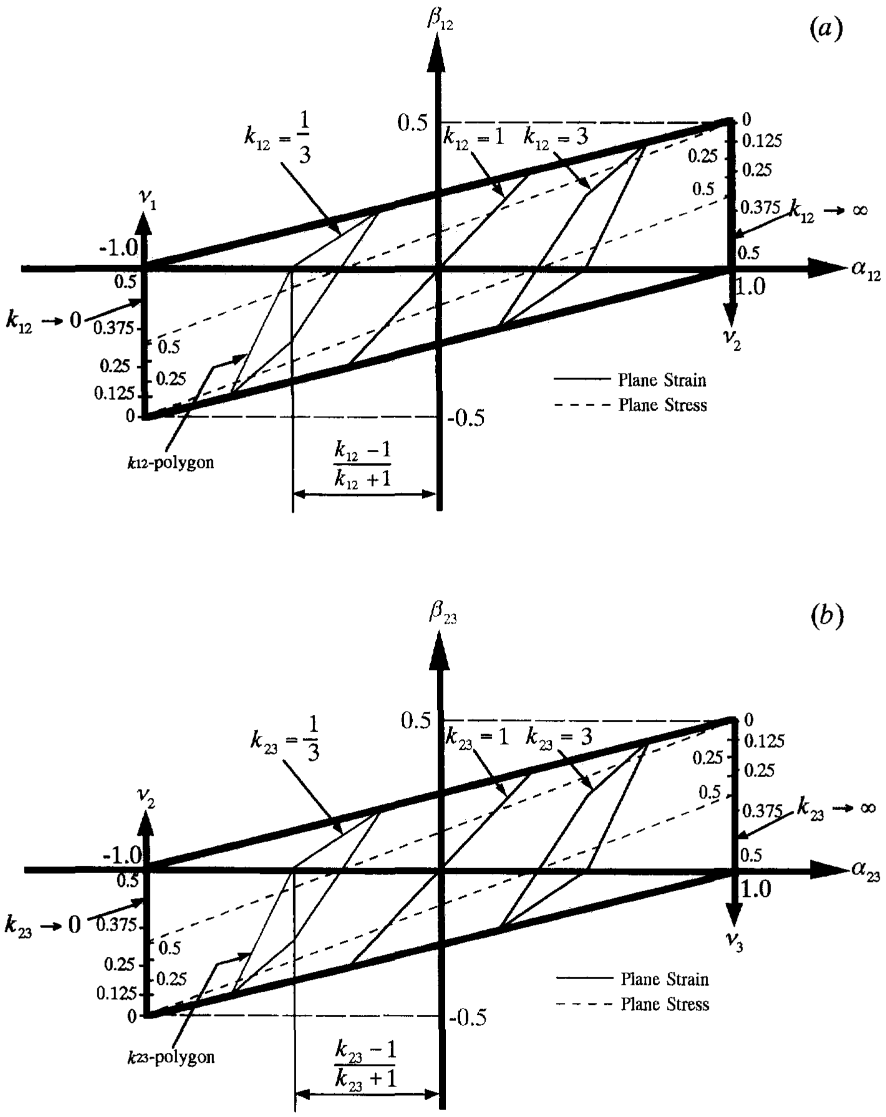


Fig. 3. Parallelogram on (a) α_{12}–β_{12} plane for materials 1 and 2, (b) α_{23}–β_{23} plane for materials 2 and 3.

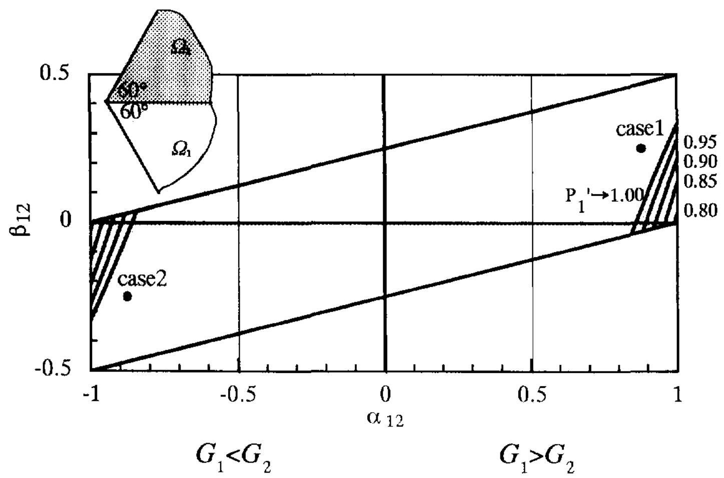


Fig. 4. Loci for roots $p = p'_1$ for two-phase bonded structure of $\varphi_1 = \varphi_2 = \pi/3$.

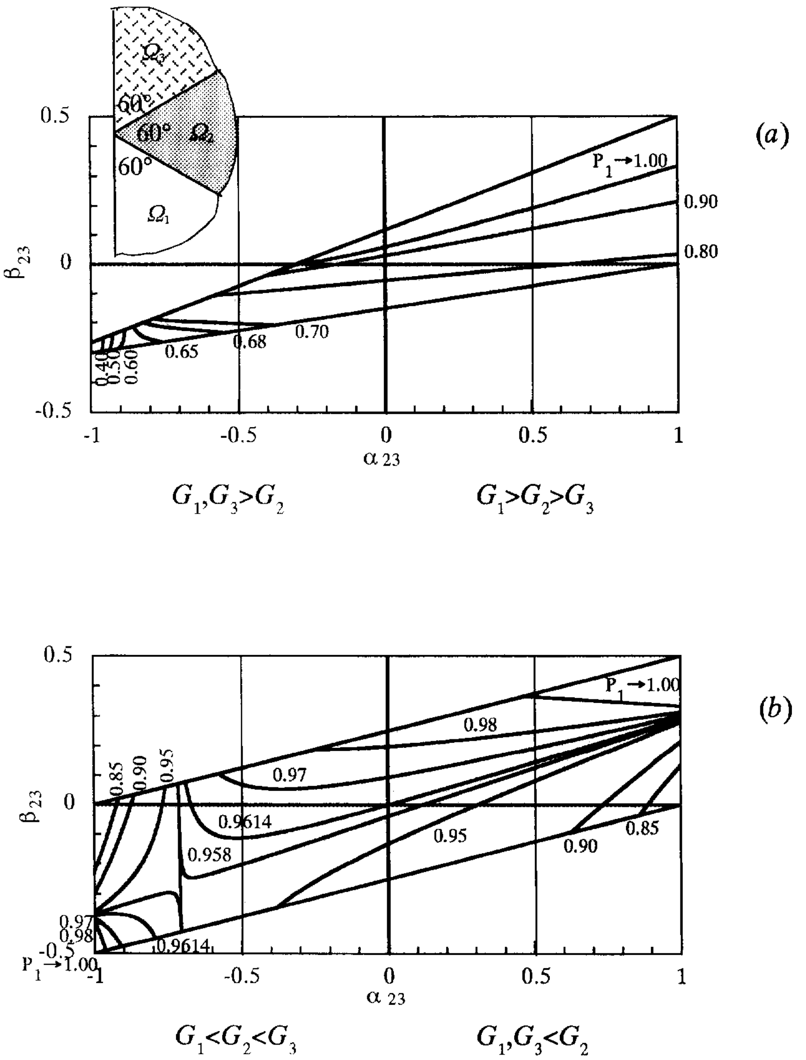


Fig. 5. Loci for roots $p = p_1$ of $S(\varphi_1, \varphi_2, \varphi_3, \alpha, \beta, -\alpha, -\beta; p_1) = 0$ for $\varphi_1 = \varphi_2 = \varphi_3 = \pi/3$ of (a) case 1, (b) case 2.

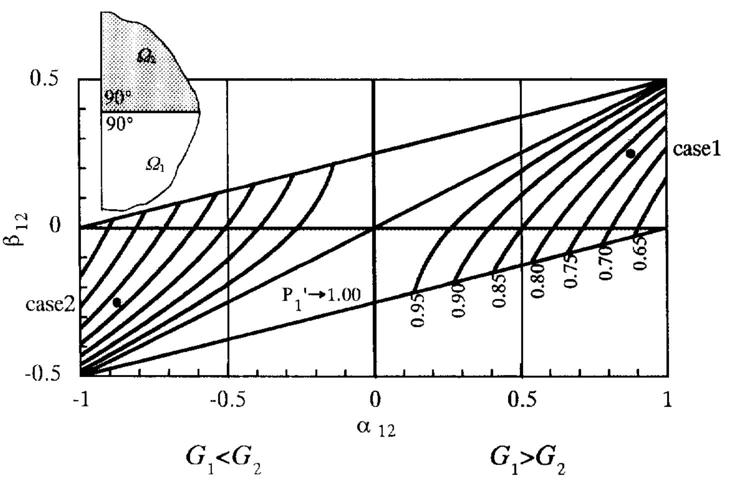
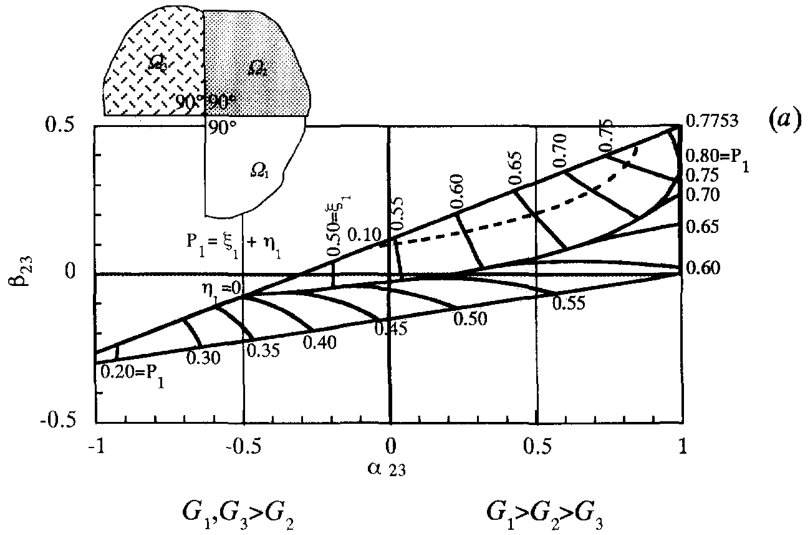


Fig. 6. Loci for roots $p = p_1$ for two-phase bonded structure of $\varphi_1 = \varphi_2 = \pi/2$.



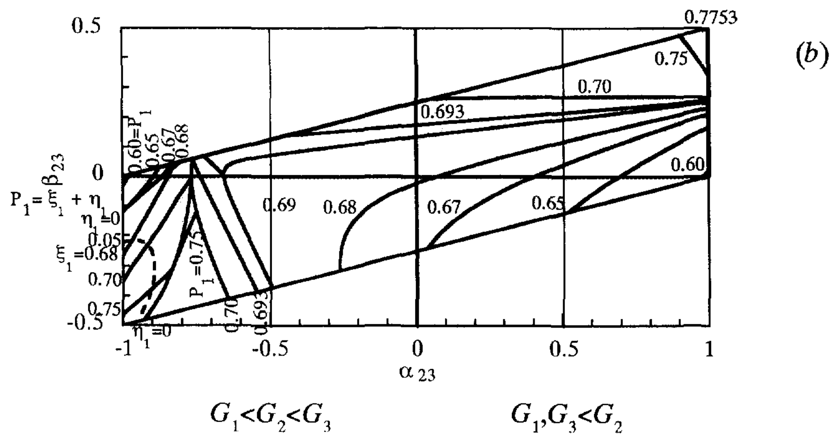


Fig. 7. Loci for roots $p = p_1$ of $S(\varphi_1, \varphi_2, \varphi_3, \alpha, \beta, -\alpha, -\beta; p_1) = 0$ for $\varphi_1 = \varphi_2 = \varphi_3 = \pi/2$ of (a) case 1, (b) case 2.

3.3. *Roots for bonded wedge with* $\varphi_1 = \varphi_3 =$ *fixed and* $\varphi_1+\varphi_2+\varphi_3 \leqslant 2\pi$

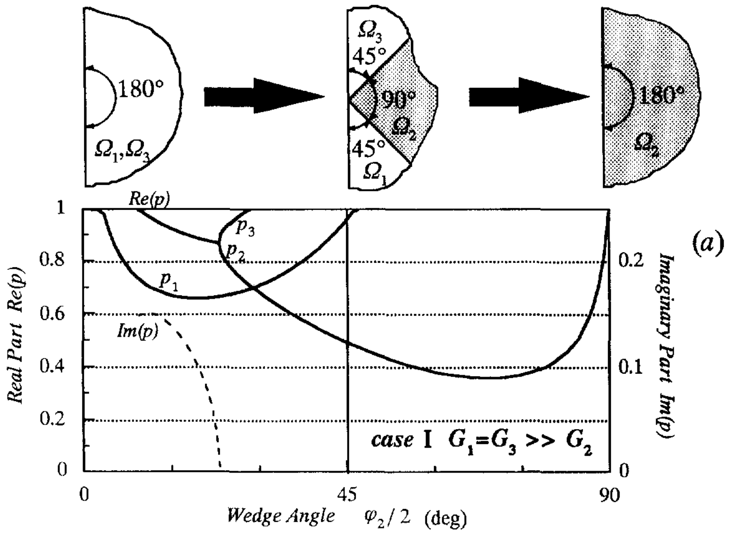
A relation between roots and φ_2 is investigated with varying the wedge angle φ_2 from 0 to 2π while holding $\varphi_1 = \varphi_3 =$ fixed. The results in wedge angle of $\varphi_1 = \varphi_3 = \pi/6$, $\pi/2$, $2\pi/3$ and $5\pi/6$ are demonstrated in Figs 11–14. The combinations of material properties are the same as used before. The left end ($\varphi_2 = 0°$) in these figures corresponds to a free–free single wedge of angle $\varphi_1+\varphi_3$ and the right end ($\varphi_1+\varphi_2+\varphi_3 = 2\pi$) to a bonded wedge occurring at a semi-infinite crack at the interface of materials 1 and 3. The roots in the condition $\varphi_2 = 0°$ agree with ones obtained from eqn (15) of Williams (1952).

4. DISCUSSION

4.1. *Loci for roots* p_1 *plotted on the* α_{23}–β_{23} *plane*

The α_{23}–β_{23} domain does not always form a parallelogram in case 1. When $(\alpha_{12}, \beta_{12}) = (0.875, 0.25)$, Poisson's ratio ν_2, from eqn (2), exists within $0.286 \leqslant \nu_2 \leqslant 0.318$ ($0.4 \leqslant \nu_2 \leqslant 0.433$ for plane stress). Hence, the values of α_{23} and β_{23} fall within a trapezoid, as shown in Figs 5(a) and 7(a).

The conditions $(\alpha_{23}, \beta_{23}) = (-0.875, -0.25)$ in case 1 and $(\alpha_{23}, \beta_{23}) = (0.875, 0.25)$ in case 2 mean that the mechanical properties of material 1 are the same as those of material



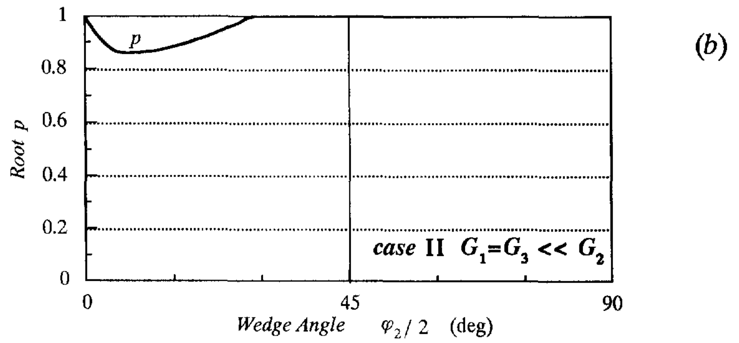


Fig. 8. Variations of roots p of $S(\varphi_1, \varphi_2, \varphi_3, \alpha, \beta, -\alpha, -\beta; p) = 0$ existing in $0 < \mathrm{Re}(p) < 1$ with varying φ_2 when a half-plane consists of bonded wedges and elastic constants have the relation (a) I—$(\alpha, \beta, -\alpha, -\beta) = (0.85, 0.30, -0.85, -0.30) - G_1 = G_3 \gg G_2$, (b) II—$(\alpha, \beta, -\alpha, -\beta) = (-0.85, -0.30, 0.85, 0.30) - G_1 = G_3 \ll G_2$.

3. The condition $(\alpha_{23}, \beta_{23}) = (0, 0)$ in each case indicates that the properties of materials 2 and 3 are identical. The roots then agree with ones obtained from eqn (19) of Bogy (1971a). Moreover, it is found from equation (6) that the condition $(\alpha_{23}, \beta_{23}) = (0.875, 0.25)$ in case 1 and $(\alpha_{23}, \beta_{23}) = (-0.875, -0.25)$ in case 2 yield the same roots p.

Case 1 corresponds to a two-phase bonded structure with the relations $0 \leqslant v_1 \leqslant 0.5$, $0.286 \leqslant v_2 \leqslant 0.318$ and $G_1 \gg G_2$, and case 2 to that with $0.286 \leqslant v_1 \leqslant 0.318$, $0 \leqslant v_2 \leqslant 0.5$ and $G_1 \ll G_2$. When $\alpha_{23} = 1$ ($k_{23} \to \infty$), roots agree with ones in the fixed–free single wedge [reported in Williams (1952)] with the angle φ_3. However, roots at $\alpha_{23} = 1$ in $\pi/2$–$\pi/2$–$\pi/2$ also yield ones in the two-phase bonded structure of materials 1 and 2 (see Fig. 7), because the following relation always holds.

$$S\left(\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \pm 0.875, \pm 0.25, 1, \beta_{23}; 0.7753\right) = 0. \tag{9}$$

It can be seen in the α_{23}–β_{23} plane that root p_1 becomes larger as $\alpha_{23} \to 1$ and smaller as $\alpha_{23} \to -1$ in case 1. That is, the order $O(r^{p-1})$ of stress singularity tends to increase when $G_1 = G_3 > G_2$, and to decrease when $G_1 = G_3 < G_2$.

As seen from Fig. 4, the stress singularity does not occur in the two-phase bonded structure ($\pi/3$–$\pi/3$) of cases 1 and 2. However, when material 3 with $\pi/3$ is bonded to the two-phase bonded structure of materials 1 and 2, the stress singularity occurs almost in the

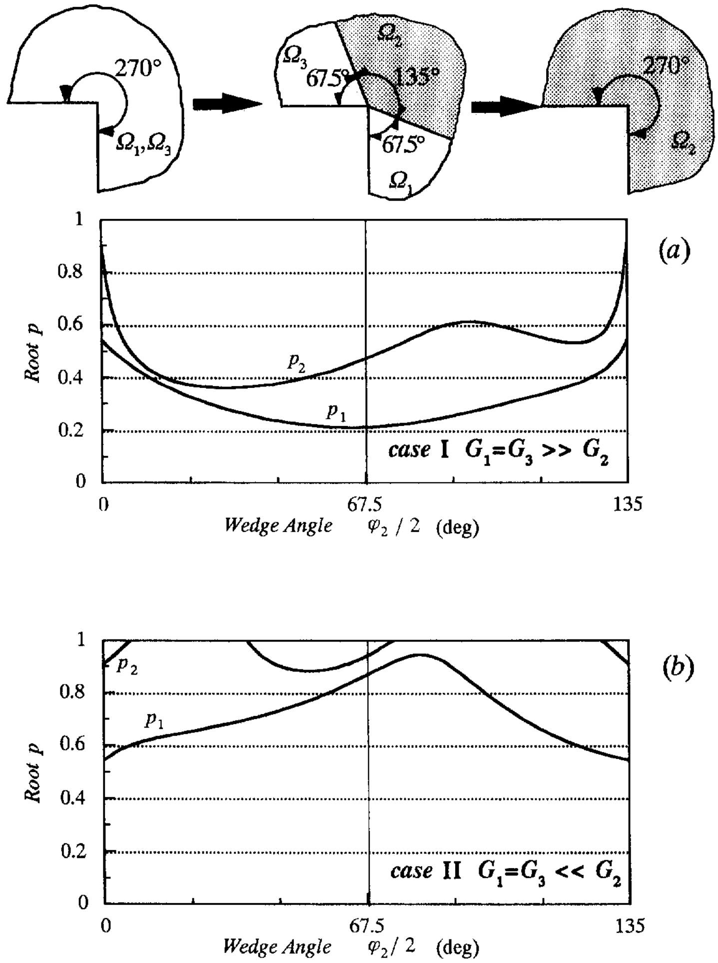


Fig. 9. Variations of roots p of $S(\varphi_1, \varphi_2, \varphi_3, \alpha, \beta, -\alpha, -\beta; p) = 0$ existing in $0 < \mathrm{Re}(p) < 1$ with varying φ_2 when a three-quarters-plane consists of bonded wedges and elastic constants have the relation (a) I—$(\alpha, \beta, -\alpha, -\beta) = (0.85, 0.30, -0.85, -0.30) - G_1 = G_3 \gg G_2$, (b) II—$(\alpha, \beta, -\alpha, -\beta) = (-0.85, -0.30, 0.85, 0.30) - G_1 = G_3 \ll G_2$.

domain of the polygon in the α_{23}–β_{23} plane (see Fig. 5). Root p_1 in the two-phase structures ($\pi/2$–$\pi/2$) corresponding to cases 1 and 2 indicated in Fig. 6 is 0.7753. Root p_1 in the three-phase structure ($\pi/2$–$\pi/2$–$\pi/2$) becomes small in comparison with 0.7753 regardless of elastic properties of material 3 (see Fig. 7). This is attributed to the increase of total angle of bonded wedges by bonding material 3. It is found from the current result, the results of Bogy (1971a), and Hein and Erdogan (1971), that the order of stress singularity becomes larger as the total angle of the bonded wedge increases. Consequently, the stress singularity in the two-phase bonded structure cannot be reduced by bonding material 3.

4.2. *Loci for roots* p *for structure with* $\varphi_1 = \varphi_3$ *and* $\varphi_1 + \varphi_2 + \varphi_3 =$ *fixed*

Root p_1 yielding the most dominant factor for the stress field in the case of $G_1 = G_3 \gg G_2$ attains a minimum at $\varphi_2/2 \cong 68.5°$ in the half-plane, at $\varphi_2/2 \cong 64°$ in the three-quarters-plane and at $\varphi_2/2 \cong 150°$ in the full-plane. Furthermore, the root p_1 attains a maximum at $\varphi_2 = 0°$ and $\varphi_1 = \varphi_3 = 0°$ in all the cases of $G_1 = G_3 \gg G_2$; the order of stress singularity yields a minimum at both ends of the range as shown in Figs 8(a)–10(a). When $G_1 = G_3 \ll G_2$, root p_1 attains a maximum at $\varphi_2/2 \cong 81.5°$ in the three-quarters-plane and at $\varphi_2/2 \cong 95°$ in the full-plane, and yields a minimum at both ends of the range as shown in Figs 8(b)–10(b). Therefore, the order $O(r^{p-1})$ of singularity in the three-phase structure becomes larger in comparison with that in a single wedge structure when the

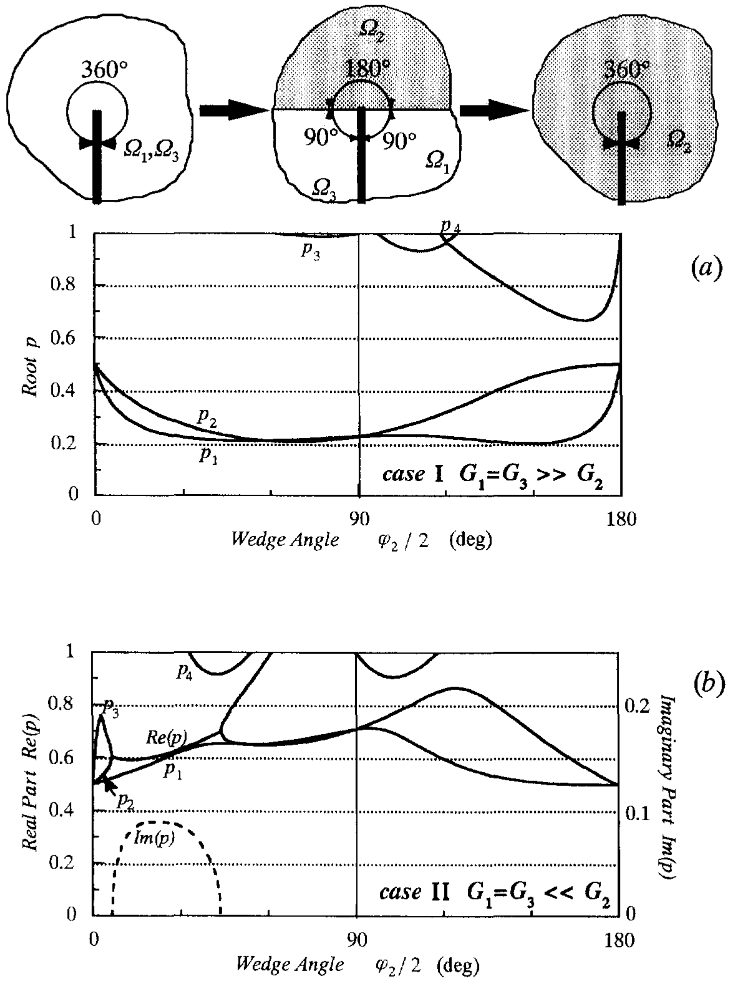


Fig. 10. Variations of roots p of $S(\varphi_1, \varphi_2, \varphi_3, \alpha, \beta, -\alpha, -\beta; p) = 0$ existing in $0 < \mathrm{Re}(p) < 1$ with varying φ_2 when a full-plane consists of bonded wedges and elastic constants have the relation (a) I—$(\alpha, \beta, -\alpha, -\beta) = (0.85, 0.30, -0.85, -0.30)$—$G_1 = G_3 \gg G_2$, (b) II—$(\alpha, \beta, -\alpha, -\beta) = (-0.85, -0.30, 0.85, 0.30)$—$G_1 = G_3 \ll G_2$.

intermediate material in the three-phase structure is softer than both side materials. On the contrary, it becomes smaller when the intermediate material is stiffer.

From the above results, we are able to make an effective way of reducing the order $O(r^{p-1})$ of the singularity by examining the relationship between the wedge angle φ_1 (or φ_3) of both side materials and roots p.

In the case where the order of stress singularity attains a maximum at $\varphi_2/2 = 68.5°$ as shown in Fig. 8(a), the variation of root p with the wedge angle φ_1 is shown in Fig. 15(a), where $\varphi_1+\varphi_2+\varphi_3$ = fixed and φ_2 = fixed. The left end ($\varphi_1 = 0°$) in Fig. 15(a) corresponds to the two-phase bonded structure with wedge angles φ_2 and φ_3, and the right end ($\varphi_3 = 0°$) with wedge angles φ_1 and φ_2. The roots then agree with the ones obtained from the eqn (19) of Bogy (1971a). It is found from Fig. 15(a) that the order $O(r^{p-1})$ becomes small with decreasing angle φ_1. At $\varphi_2/2 \cong 64°$ and 150° yielding a maximum order $O(r^{p-1})$ shown in Figs 9(a) and 10(a), the order $O(r^{p-1})$ behaves like the result shown in Fig. 15(a) with decreasing angle φ_1. In the case where the order of singularity attains a minimum at $\varphi_2/2 = 81.5°$ as shown in Fig. 9(b), the variation of root p with the wedge angle φ_1 is shown in Fig. 15(b). It is found from Fig. 15(b) that the order $O(r^{p-1})$ becomes large with decreasing the angle φ_1. At $\varphi_2/2 \cong 95°$ yielding a minimum order $O(r^{p-1})$ shown in Fig. 10(b), the order $O(r^{p-1})$ behaves like the result shown in Fig. 15(b) with decreasing angle

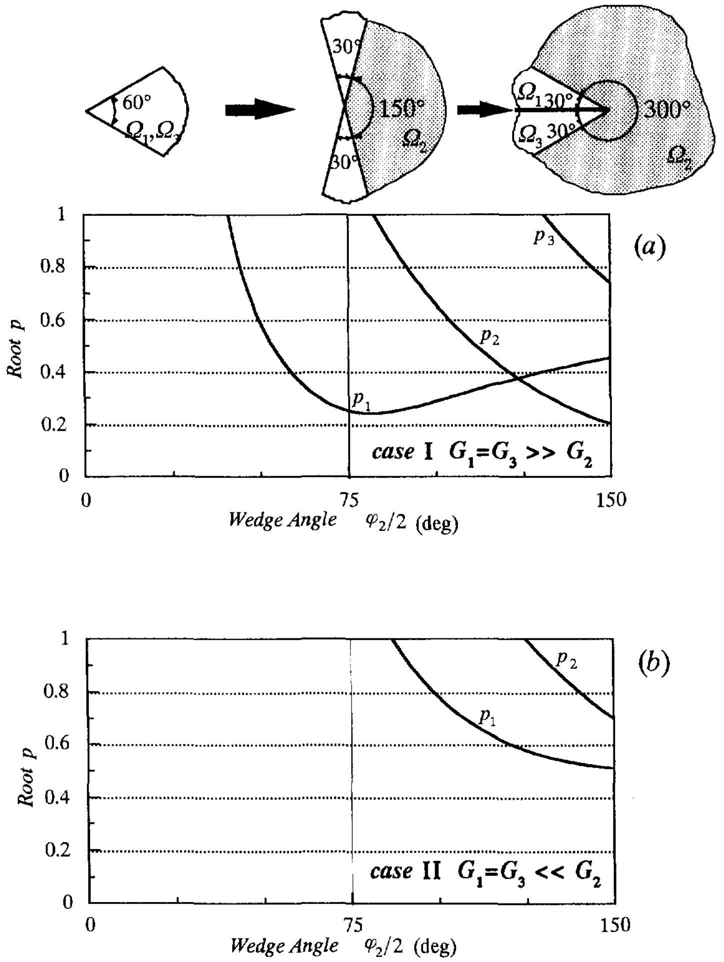


Fig. 11. Variations of roots p of $S(\varphi_1, \varphi_2, \varphi_3, \alpha, \beta, -\alpha, -\beta : p) = 0$ existing in $0 < \mathrm{Re}(p) < 1$ with varying φ_2 when $\varphi_1 = \varphi_3 = \pi/6$. The relation of elastic constants is (a) I—$(\alpha, \beta, -\alpha, -\beta) = (0.85, 0.30, -0.85, -0.30) - G_1 = G_3 \gg G_2$, (b) II—$(\alpha, \beta, -\alpha, -\beta) = (-0.85, -0.30, 0.85, 0.30) - G_1 = G_3 \ll G_2$.

φ_1. Consequently, it is found that the order of singularity can be reduced by not equalizing φ_1 with φ_3 when $G_1 = G_3 \gg G_2$, and equalizing φ_1 with φ_3 when $G_1 = G_3 \ll G_2$.

When $G_1 = G_3 \ll G_2$, root p_1 in the half-plane attains a minimum at $\varphi_2/2 \cong 8^\circ$ and the stress singularity disappears between the angle, $\varphi_2/2$, of 28° and 90° (see Fig. 8b). When the order $O(r^{p-1})$ yields a maximum at $\varphi_2/2 = 8^\circ$ as shown in Fig. 8(b), the relation between the wedge angle φ_1 and root p is as shown in Fig. 16.

4.3. *Loci for roots* p *for structure with* $\varphi_1 = \varphi_3 =$ *fixed*, $\varphi_1 + \varphi_2 + \varphi_3 \leqslant 2\pi$

In the case of $\varphi_1 = \varphi_3 = \pi/6$ shown in Fig. 11, when $G_1 = G_3 \gg G_2$, root p_1 appears over $\varphi_1 + \varphi_2 + \varphi_3 = 142^\circ$ ($\varphi_2/2 = 41^\circ$), when $G_1 = G_3 \ll G_2$, it appears over $\varphi_1 + \varphi_2 + \varphi_3 = 234^\circ$ ($\varphi_2/2 = 87^\circ$). Also, the order $O(r^{p-1})$ of singularity tends to become larger as the wedge angle φ_2 increases. In the case of $\varphi_1 = \varphi_3 = \pi/2$ shown in Fig. 12, when $G_1 = G_3 \gg G_2$, the second root p_2 appears over $\varphi_2/2 = 3.05^\circ$, when $G_1 = G_3 \ll G_2$, it appears over $\varphi_2/2 = 38.9^\circ$. This root p_2 becomes a double root of $p_2 = 2.00$ at $\varphi_2 = 0^\circ$, i.e. a free–free single half plane. Hence, when $G_1 = G_3 \gg G_2$, it is found that the root p_2 rapidly varies with the wedge angle φ_2. Moreover, root p_1 attains a minimum at $\varphi_2/2 = 62.5^\circ$, and becomes a double root at $\varphi_2/2 = 90^\circ$. Also, when $G_1 = G_3 \ll G_2$, root p_1 gradually varies with φ_2, and becomes a double root at $\varphi_2/2 = 90^\circ$. The roots in case of $\varphi_2/2 = 90^\circ$ agree with ones obtained from eqn (28) of Bogy (1971b). In the case of $\varphi_1 = \varphi_3 = 2\pi/3$ shown in Fig. 13, the second root p_2 has no singularity ($p_2 = 1.1489$) at $\varphi_2 = 0^\circ$, and it has a singularity over $\varphi_2/2 = 0.675^\circ$

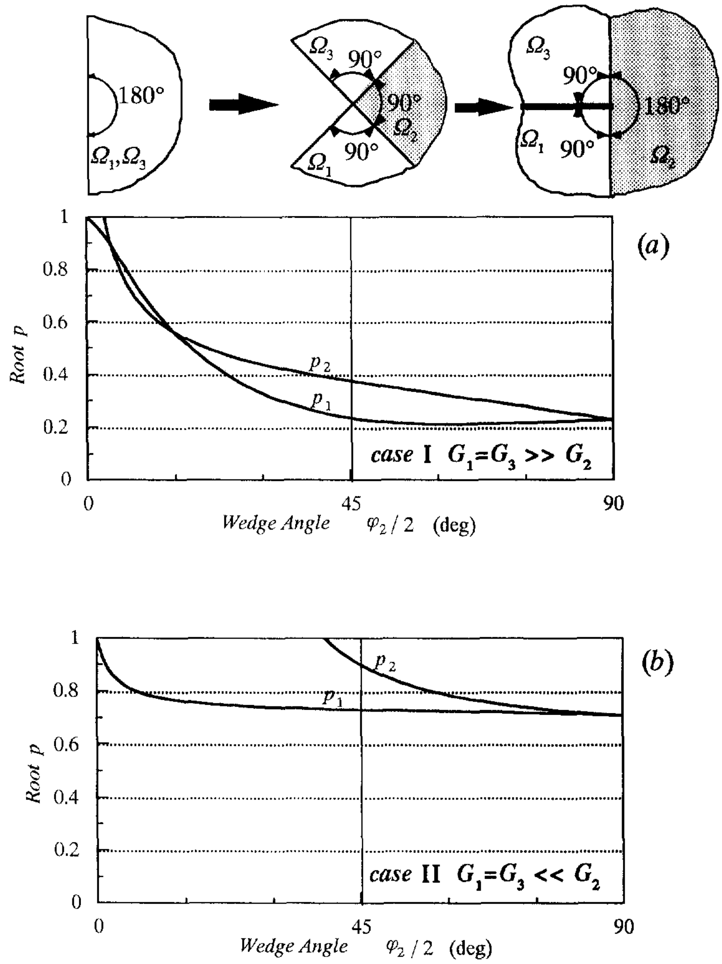


Fig. 12. Variations of roots p of $S(\varphi_1, \varphi_2, \varphi_3, \alpha, \beta, -\alpha, -\beta; p) = 0$ existing in $0 < \mathrm{Re}(p) < 1$ with varying φ_2 when $\varphi_1 = \varphi_3 = \pi/2$. The relation of elastic constants is (a) I—$(\alpha, \beta, -\alpha, -\beta) = (0.85, 0.30, -0.85, -0.30)$–$G_1 = G_3 \gg G_2$, (b) II—$(\alpha, \beta, -\alpha, -\beta) = (-0.85, -0.30, 0.85, 0.30)$–$G_1 = G_3 \ll G_2$.

when $G_1 = G_3 \gg G_2$ and over $\varphi_2/2 = 30.55°$ when $G_1 = G_3 \ll G_2$. Also, when $G_1 = G_3 \ll G_2$, the order $O(r^{p_1-1})$ of singularity of root p_1 becomes a smaller as the wedge angle φ_2 increases. For $\varphi_1 = \varphi_3 = 5\pi/6$ shown in Fig. 14, when $G_1 = G_3 \ll G_2$, the order $O(r^{p_1-1})$ of root p_1 decreases as the wedge angle φ_2 increases in contrast to the results illustrated in Figs 11, 12, 13(a) and 14(a). Moreover, the third root p_3 then appears over $\varphi_2/2 \cong 3°$. The roots p_1 and p_3 combine at $\varphi_2/2 \cong 16°$ and become a complex root.

In almost all of these results, it is found that the order $O(r^{p-1})$ of singularity becomes larger as the wedge angle φ_2 increases. However, when the wedge angles of both side materials are large in $G_1 = G_3 \ll G_2$, the order becomes smaller as the angle φ_2 increases. In the case where a crack exists in a stiffer material indicated by (a) in Figs 11–14, the smallest root varies little with the wedge angles of both side materials (its root is 0.2017 when $\varphi_1 = \varphi_3 = \pi/6$, 0.2286 when $\pi/2$, 0.2110 when $2\pi/3$ and 0.2250 when $5\pi/6$). That is, when $G_1 = G_3 \gg G_2$ and $\varphi_1+\varphi_2+\varphi_3 = 2\pi$, the difference of pairs of the wedge angles does not strongly influence the order of stress singularity in comparison with that of the mechanical properties. On the contrary, in the case (indicated by (b)) where a crack exists in a softer material, the smallest root varies with the wedge angles of both side materials (its root is 0.5111 when $\varphi_1 = \varphi_3 = \pi/6$, 0.7082 when $\pi/2$, 0.6470 when $2\pi/3$ and 0.6216 when $5\pi/6$). Furthermore, it is seen from results of $\varphi_1+\varphi_2+\varphi_3 = 2\pi$ that the order $O(r^{p-1})$ of singularity at the tip of a crack existing in the stiff side of two-phase materials becomes large in comparison with that in soft side. This result agrees with the one well known from

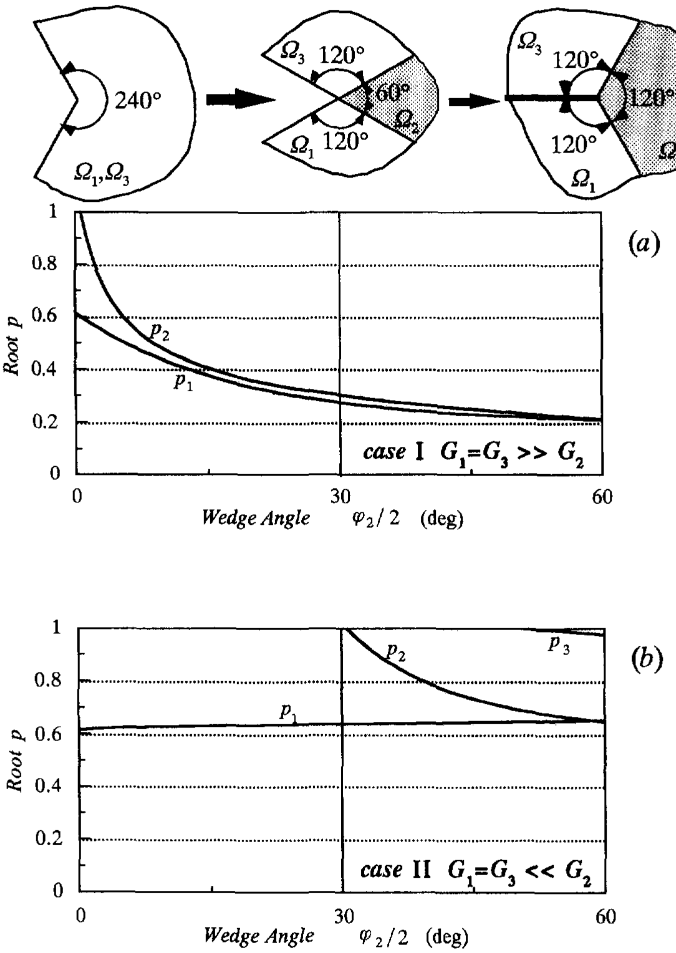


Fig. 13. Variations of roots p of $S(\varphi_1, \varphi_2, \varphi_3, \alpha, \beta, -\alpha, -\beta; p) = 0$ existing in $0 < \mathrm{Re}(p) < 1$ with varying φ_2 when $\varphi_1 = \varphi_3 = 2\pi/3$. The relation of elastic constants is (a) I—$(\alpha, \beta, -\alpha, -\beta) = (0.85, 0.30, -0.85, -0.30)$—$G_1 = G_3 \gg G_2$, (b) II—$(\alpha, \beta, -\alpha, -\beta) = (-0.85, -0.30, 0.85, 0.30)$—$G_1 = G_3 \ll G_2$.

previous investigations on a crack intersecting with a bimaterial interface (Bogy, 1971b; Cook and Erdogan 1972; Fenner 1976; Barsoum, 1988).

4.4. *Method reducing stress singularity in a two-phase structure*

The method reducing the stress singularity in a two-phase bonded structure is presented now. Considering two copper–alumina composites with the wedge angles (type 1 : $\pi/2$–$\pi/2$, type 2 : $\pi/2$–π) and the elastic properties shown in Fig. 17 as an example of the two-phase bonded structure, the method reducing the stress singularity on the basis of plane strain analysis is discussed. We know from the results of this study that in order to lessen the stress singularity, it is necessary for the intermediate region to be made from a stiffer material. Now, the two methods considered to reduce the stress singularity are presented: method (A) where the third material with softer property in comparison with alumina is bonded to the free surface of alumina in the copper–alumina composite, and method (B) where copper is divided into two wedges with the appropriate angles and each divided wedge is bonded to the both sides of alumina (see Fig. 1).

In method (A), materials with elastic properties such as epoxy and steel shown in Fig. 17 are employed as the third material, and the relationship between the wedge angle φ_3 of those materials and roots is shown in Fig. 18. It is found that root p_1 holds almost constant, even if the angle φ_3 varies in the case of the copper–alumina composite with wedge angles of type 2 ($\pi/2$–π). Also, roots p_1, p_2 and p_3 vary little with the angle φ_3 within a range of

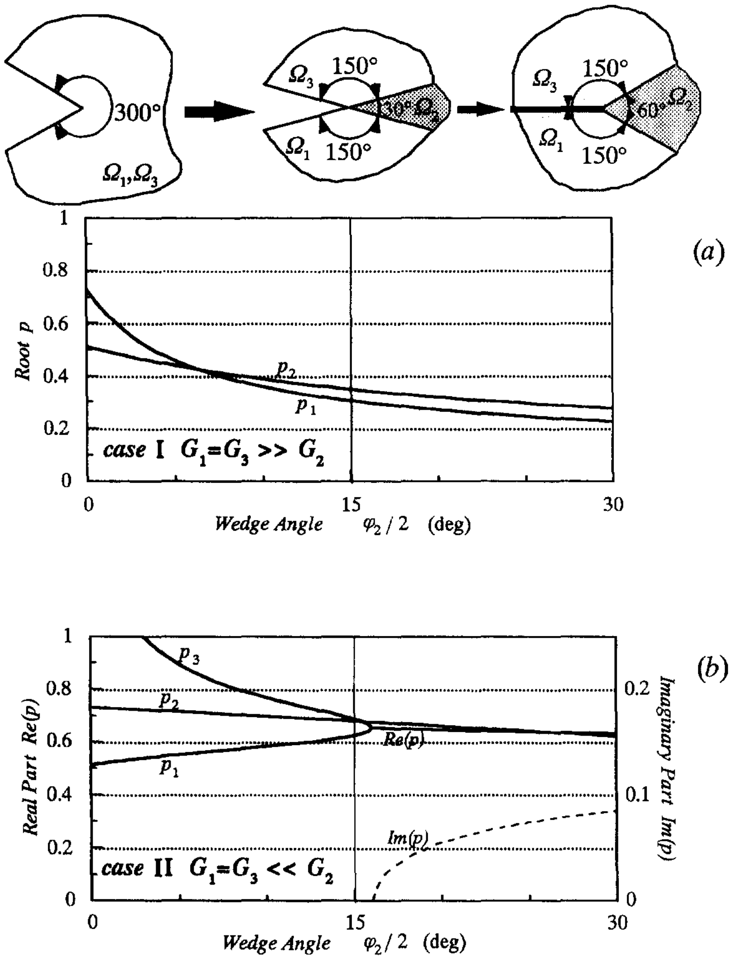


Fig. 14. Variations of roots p of $\mathbf{S}(\varphi_1, \varphi_2, \varphi_3, \alpha, \beta, -\alpha, -\beta; p) = 0$ existing in $0 < \mathrm{Re}(p) < 1$ with varying φ_2 when $\varphi_1 = \varphi_3 = 5\pi/6$. The relation of elastic constants is (a) I—$(\alpha, \beta, -\alpha, -\beta) = (0.85, 0.30, -0.85, -0.30)-G_1 = G_3 \gg G_2$, (b) II—$(\alpha, \beta, -\alpha, -\beta) = (-0.85, -0.30, 0.85, 0.30)-G_1 = G_3 \ll G_2$.

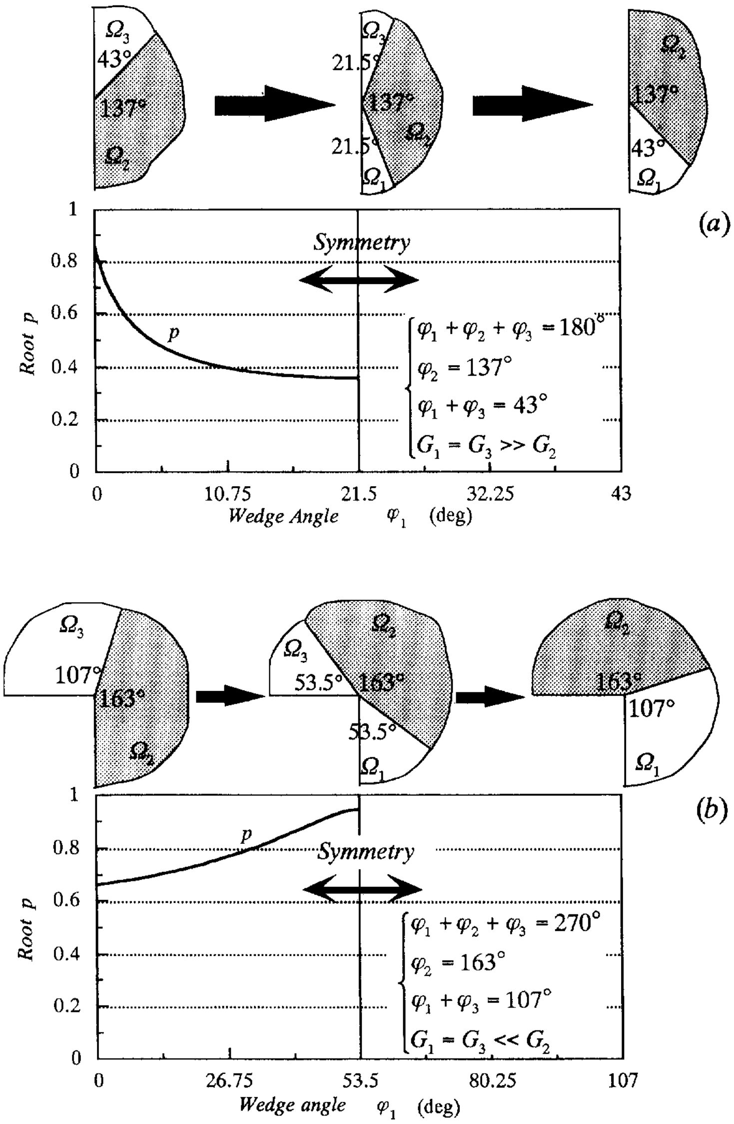


Fig. 15. Relation between roots p and wedge angle φ_1 of region Ω_1 in $\varphi_1+\varphi_2+\varphi_3$ = fixed, (a) when $G_1 = G_3 \gg G_2$, $\varphi_2 = 137°$ and $\varphi_1+\varphi_3 = 43°$, (b) when $G_1 = G_3 \ll G_2$, $\varphi_2 = 163°$ and $\varphi_1+\varphi_3 = 107°$.

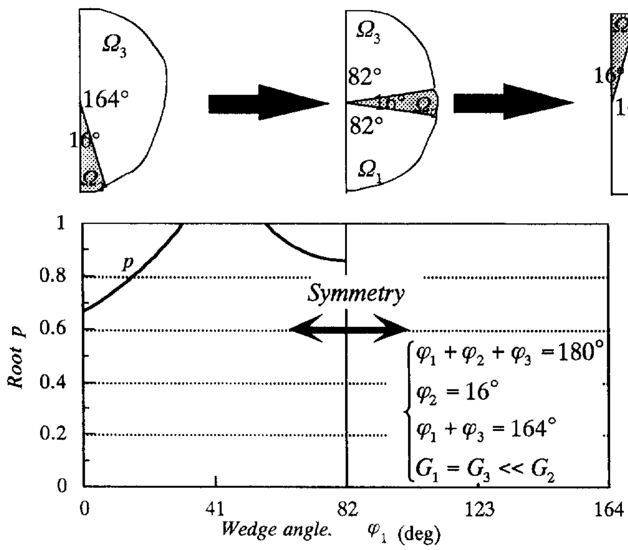


Fig. 16. Relation between roots p and wedge angle φ_1 of region Ω_1 in $\varphi_1+\varphi_2+\varphi_3$ = fixed, when $G_1 = G_3 \ll G_2$, $\varphi_2 = 16°$ and $\varphi_1+\varphi_3 = 164°$.

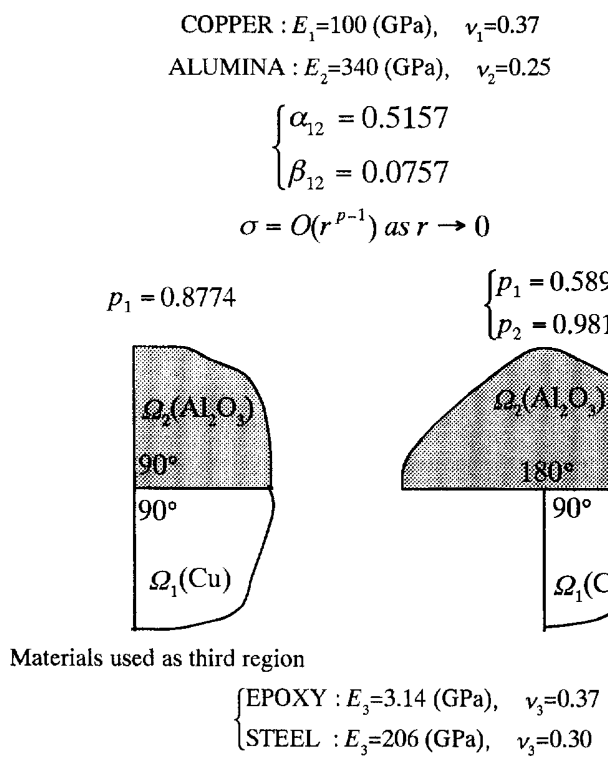


Fig. 17. Two models for copper–alumina composite.

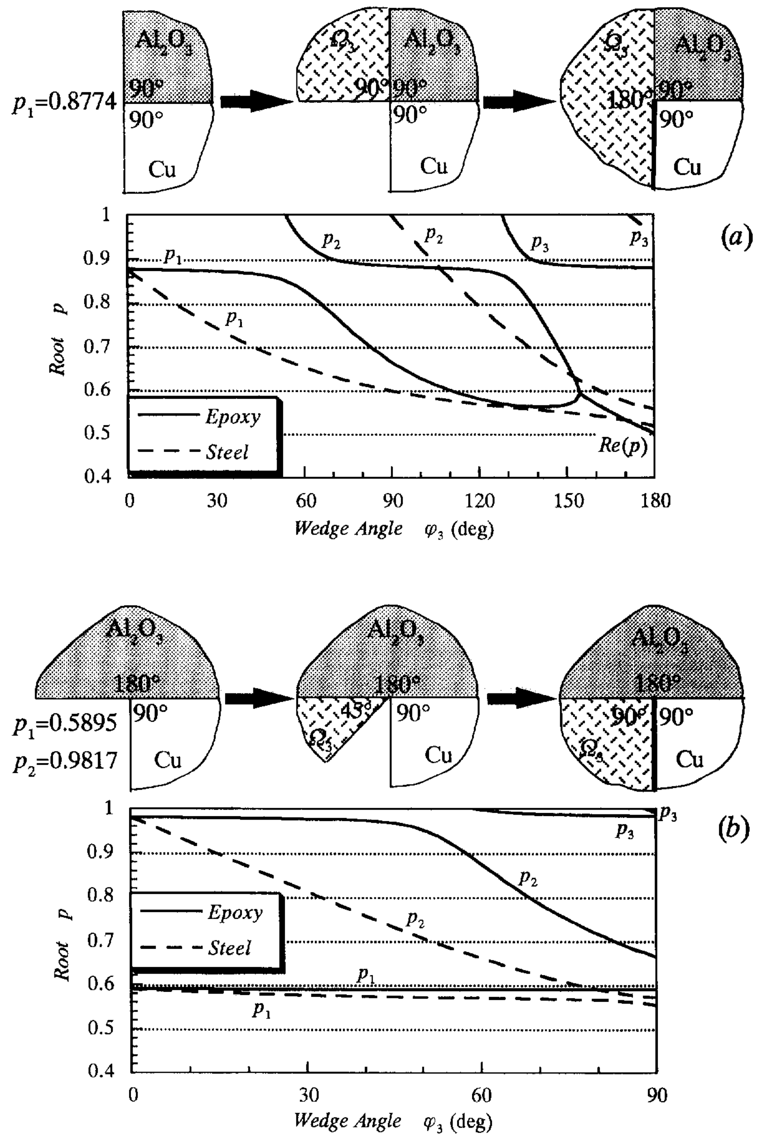


Fig. 18. Relation between roots p and wedge angle φ_3 of third material bonded to alumina, (a) when $\varphi_1 = \varphi_2 = \pi/2$, (b) when $\varphi_1 = \pi/2$ and $\varphi_2 = \pi$.

angles: $0^\circ \leqslant \varphi_3 \leqslant 45^\circ$ for p_1, $85^\circ \leqslant \varphi_3 \leqslant 110^\circ$ for p_2 and $150^\circ \leqslant \varphi_3 \leqslant 180^\circ$ for p_3 in type 1 and $0^\circ \leqslant \varphi_3 \leqslant 40^\circ$ for p_2 and $70^\circ \leqslant \varphi_3 \leqslant 90^\circ$ for p_3 in type 2 when epoxy is bonded as the third material. However, the stress singularity does not then decrease. Furthermore, the second and third roots, p_2 and p_3, appear in the composite with wedge angles of type 1, and the third root p_3 appears in that of type 2. Hence, this is not an effective method for reducing the stress singularity.

The relationship in method (B) between the wedge angle φ_3 and roots is shown in Fig. 19, where the regions occupied by alumina and copper are represented as $\Omega_2(\varphi_2 = \pi/2$ and $\pi)$ and as $\Omega_1+\Omega_3(\varphi_1+\varphi_3 = \pi/2)$, respectively. In the case of a composite structure of type 1, the stress singularity becomes small as the wedge angle φ_3 increases and disappears in the range of $29^\circ \leqslant \varphi_3 \leqslant 61^\circ$ as shown in Fig. 19(a). Furthermore, a similar tendency is seen in the case of composite structure of type 2. Consequently, method (B) is effective for reducing the stress singularity in the two-phase structure.

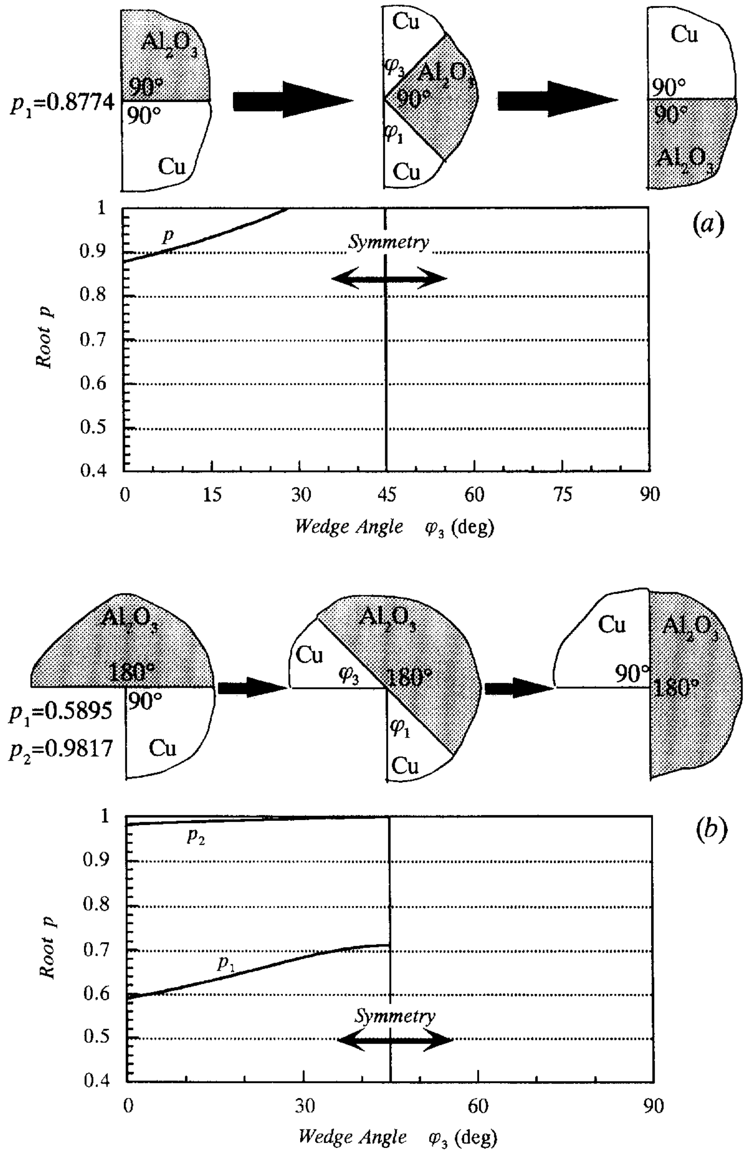


Fig. 19. Relation between roots p and wedge angle φ_3 of region Ω_3 in $\varphi_1+\varphi_2+\varphi_3$ = fixed, (a) when $\varphi_2 = \pi/2$ and $\varphi_1+\varphi_3 = \pi/2$, (b) when $\varphi_2 = \pi$ and $\varphi_1+\varphi_3 = \pi/2$.

5. CONCLUSIONS

The order of stress singularity in the three-phase bonded structures of $\pi/3$–$\pi/3$–$\pi/3$ and $\pi/2$–$\pi/2$–$\pi/2$ was investigated on the basis of two-dimensional elasticity. Variations of the order of singularity with material properties were shown on the α_{23}–β_{23} plane. The order $O(r^{p-1})$ of stress singularity tends to increase when the intermediate material in the three-phase structure is softer than the both side materials, and vice versa. Under the condition that materials 1 and 3 are identical, the relation between the wedge angle φ_2 of the intermediate material and the order of singularity was investigated in two cases of $\varphi_1+\varphi_2+\varphi_3$ = fixed and $\varphi_1 = \varphi_3$ = fixed. Also, a relation between the wedge angles of both side materials and the order of singularity was examined under $\varphi_1+\varphi_2+\varphi_3$ = fixed and φ_2 = fixed. It is necessary for reducing the order of singularity not to equalize φ_1 and φ_3 when $G_1 = G_3 \gg G_2$, and to equalize φ_1 with φ_3 when $G_1 = G_3 \ll G_2$. In the case of $G_1 = G_3 \gg G_2$, $\varphi_1+\varphi_2+\varphi_3 = 2\pi$ and $\varphi_1 = \varphi_3$, i.e. when a crack exists in a stiff material, the smallest root yielding the most dominant factor for the stress field at the tip of crack varies little with the wedge angles of both side materials. The results deduced in the present paper agree well with results reported by other investigators. Moreover, in the case where the total angle of the bonded wedge and mechanical properties for each material are fixed, the way to reduce the stress concentration at the apex in the two-phase structure is presented.

REFERENCES

Barsoum, R. S. (1988). Theoretical basis of the finite element iterative method for the eigenvalue problem in stationary cracks. *Int. J. Numer. Meth. Engng* **26**, 531–539.

Bogy, D. B. (1968). Edge-bonded dissimilar orthogonal elastic wedges under normal and shear loading. *J. Appl. Mech.* **35**, 460–466.

Bogy, D. B. (1970). On the problem of edge-bonded elastic quarter-planes loaded at the boundary. *Int. J. Solids Structures* **6**, 1287–1313.

Bogy, D. B. (1971a). Two-edge bonded elastic wedges of different materials and wedge angles under surface tractions. *J. Appl. Mech.* **38**, 377–386.

Bogy, D. B. (1971b). On the plane elastostatic problem of a loaded crack terminating at a material interface. *J. Appl. Mech.* **38**, 911–918.

Cook, T. S. and Erdogan, F. (1972). Stresses in bonded materials with a crack perpendicular to the interface. *Int. J. Engng Sci.* **10**, 677–697.

Davis, B. (1984). *Integral Transforms and Their Applications*. Springer, Berlin.

Dempsey, J. P. and Sinclair, G. B. (1979). On the stress singularities in the plane elasticity of the composite wedge. *J. Elasticity* **9**, 373–391.

Dundurs, J. (1969). Discussion of edge-bonded dissimilar orthogonal elastic wedges under normal and shear loading. *J. Appl. Mech.* **36**, 650–652.

Fenner, D. N. (1976). Stress singularities in composite materials with an arbitrarily oriented crack meeting an interface. *Int. J. Fracture Mech.* **12**, 705–721.

Hein, V. L. and Erdogan, F. (1971). Stress singularities in a two-material wedge. *Int. J. Fracture Mech.* **7**, 317–330.

Koguchi, H., Inoue, T. and Yada, T. (1995). Stress singularity in three-phase bonded structure. *J. Appl. Mech.* (in press).

Pageau, S. S., Joseph, P. F. and Biggers, S. B. (1994). The order of stress singularities for bonded and disbonded three-material junctions. *Int. J. Solids Structures* **31**, 2979–2997.

Sneddon, I. N. (1951). *The Use of Integral Transforms*, McGraw-Hill, New York.

Theocaris, P. S. (1974). The order of singularity at a multi-wedge corner of a composite plate. *Int. J. Engng Sci.* **12**, 107–120.

Williams, M. L. (1952). Stress singularities resulting from various boundary conditions in angular corners of plates in extension. *J. Appl. Mech.* **19**, 526–528.